A line-and-conic theorem having a visual correlate*

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This article describes a new and general line-and-conic property\(^1\), leading to a consideration of a ‘six-point’ circle and the related geometry of an interesting visual illusion.

THEOREM: The sum of the two angles included by the tangents to any conic from the ends of any straight line, is equal to the sum of the angles subtended by the line at the foci.

Figure 1:

\(\alpha + \beta = \varepsilon_1 + \varepsilon_2\) (see Figure 1).

As \(CS_1\) and \(DS_1\) bisect angles \(ES_1F\) and \(GS_1H\) respectively, let \(ES_1F = 2\alpha_1\), and \(GS_1H = 2\beta_1\). Then, as \(AS_1\) and \(BS_1\) bisect angles \(ES_1G\) and \(FS_1H\) respectively, it follows fairly easily that

\[\varepsilon_1 = \alpha_1 + \beta_1,\]

\(^1\)This revision (May 2005) of the original article incorporates some typographic corrections, footnotes, and minor improvements to Figure 2.

and, with similar labelling for $S_2$, $\varepsilon_2 = \alpha_2 + \beta_2$. Therefore
\[ \varepsilon_1 + \varepsilon_2 = (\alpha_1 + \alpha_2) + (\beta_1 + \beta_2). \]
By considering the triangle pairs $CFS_1$, $CFS_2$ and $DGS_1$, $DGS_2$, it can be shown that
\[ \alpha = \alpha_1 + \alpha_2 \quad \text{and} \quad \beta = \beta_1 + \beta_2 \]
from which it follows, as required, that
\[ \varepsilon_1 + \varepsilon_2 = \alpha + \beta. \]
(Although the sign of the angles in equation 1 clearly varies according to the orientation of line and conic, a simple sign convention allows the relative signs to be readily determined.)

In the case of the parabola, as one of the foci is at infinity, one of the angles at the foci must be zero. The equation therefore reduces to $\varepsilon = \alpha + \beta$. In the case of the circle, let $\varepsilon_1 = \varepsilon_2 = \varepsilon$. The equation then reduces to $2\varepsilon = \alpha + \beta$.

![Diagram](https://via.placeholder.com/150)

Figure 2:

Now applying this theorem to two circles facilitates the following ‘six-point’ circle proof (see Figure 2). Let $O$ and $O'$ be the centres of circles whose respective radii are $r$ and $r'$. Let the internal and external tangents intersect in $A, A', B, B'$. As $AB$ and $A'B'$ are perpendicular to $OO'$, then $AA'B = AB'B$. It follows from the above result that $AA'B + AB'B = 2AO'B$ and therefore $AA'B = AB'B = AO'B$. Similarly $A'AB' = A'BB' = A'OB'$ and so the two centres $O$ and $O'$ lie on the circle $A, A', B, B'$. Therefore the six points $A, A', B, B', O, O'$ are concyclic. Furthermore, if $AB = 2d$ and $A'B' = 2d'$, it follows that $rr' = dd'$. The details of the proof of this result are left to the interested reader.

Finally, the following locus property which has an interesting visual correlate, follows directly from the above results. Consider the circle $ABO$ in Figure 3, where $M$ is the midpoint of $AB$, and $AB \perp MO$. 
THEOREM. If $PP'$ is a variable chord of any circle ($C_1$) subtending the same constant angle at the centre ($O$) as that subtended by $AB$ at $O$, then the locus ($I$) of the intersection of $AP$ and $BP'$ is an arc of the circle $ABO$.

PROOF. Construct any circle ($C_2$) smaller than and concentric with $C_1$ and let tangents from $A$ and $B$ to the circle $C_2$ intersect in $I$ and cut $C_1$ in $P$ and $P'$, all as shown in Figure 3. As an easy consequence of of the earlier result, angles $AIB$ and $AOB$ are equal and so $I$ lies on the circle $ABO$. By the same argument, $POP' = PIP'$, and therefore $POP' = AOB$. From this it follows that $QOQ' = POP'$, and therefore the intersection $I$ lies on the circle $ABO$ (irrespective of whether the chord is at $PP'$ or $QQ'$). This holds for all circles $C_2$, and so the locus $I$ is an arc of the circle $ABO$ for all positions of the chord $PP'$.

Furthermore, since $P, I, O, P'$ are concyclic (as $PIP' = POP'$) it follows that $IPO = IP'O$. Thus the lines $AP$ and $BP'$ meet the circle $C_1$ with the same angle of incidence $\gamma$.

Interestingly, the extent of the locus depends on the position of the chord $AB$ in relation to the circle $C_1$, being a simple arc if $AB$ is outside $C_1$. However, if $AB$ is a chord of $C_1$, then one revolution of $PP'$ results in $I$ making one complete revolution of $ABO$. If $AB$ is inside $C_1$, then the locus is two revolutions of $ABO$.

A visual correlate (the Pulfrich effect)

Under certain circumstances it is possible to actually see the locus of the point $I$ as the path of an object which appears to move back and forth along an arc of the circle $ABO$, in a striking visual illusion.

Consider the eyes to be at the points $A$ and $B$, viewing an object $P$ rotating...
clockwise with constant angular velocity (ω) in a horizontal plane about a centre O. If P is viewed horizontally and binocularly with a neutral density filter in front of the right eye (B), and with AB perpendicular to MO, then under certain conditions of viewing distance (MO), angular velocity (ω), and filter density (i.e. when POP′ = AOB), the object P will seem not to be rotating about O, but will appear to be moving back and forth from side to side, passing through the centre O and describing an arc of the circle ABO.

The illusion is due to the fact that the filter introduces a slight visual time delay (say, t) which is therefore equivalent to the angle POP′ (= ωt). The filtered eye therefore, always sees the object where it was a few milliseconds ago (i.e. at P′). It is thought that fusion of these two images results in the object P appearing to be at the position I.

This illusion is a particular manifestation of a little known phenomenon called the Pulfrich effect\(^2\), after Carl Pulfrich (1858–1927) who first investigated it at the Carl Zeiss laboratories in Jena. The geometry of this circular configuration gives an unusual way of considering this effect.

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