A line-and-conic theorem having a visual correlate*

R.W.D. Nickalls
Department of Anaesthesia, City Hospital, Nottingham, UK
dicknickalls@compuserve.com

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This article describes a new and general line-and-conic property[1], leading to a consideration of a 'six-point' circle and the related geometry of an interesting visual illusion.

THEOREM: *The sum of the two angles included by the tangents to any conic from the ends of any straight line, is equal to the sum of the angles subtended by the line at the foci.*

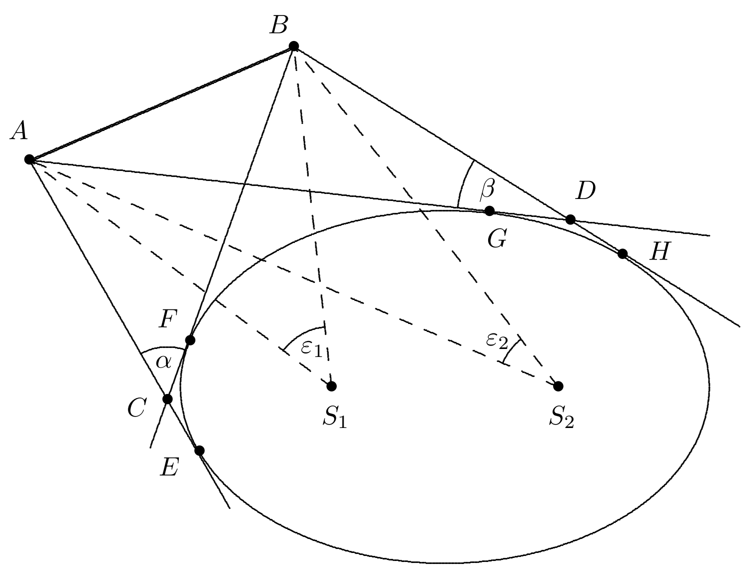

Figure 1:

TO PROVE: $\alpha + \beta \;=\; \varepsilon_1 + \varepsilon_2$ (see Figure 1).

As CS_1 and DS_1 bisect angles ES_1F and GS_1H respectively, let $ES_1F = 2\alpha_1$, and $GS_1H = 2\beta_1$. Then, as AS_1 and BS_1 bisect angles ES_1G and FS_1H respectively, it follows fairly easily that

$$\varepsilon_1 = \alpha_1 + \beta_1,$$

*This revision (May 2005) of the original article incorporates some typographic corrections, footnotes, and minor improvements to Figure 2.

[1]see also: Nickalls RWD (2000). A conic theorem generalised: directed angles and applications. *The Mathematical Gazette*, 84 (July), 232–241. `http://www.m-a.org.uk/docs/library/2068.pdf` Rigby J (2002). A proof of Nickalls' theorem on tangents and foci of a conic. *The Mathematical Gazette*, 86, 322–324.

and, with similar labelling for S_2, $\varepsilon_2 = \alpha_2 + \beta_2$. Therefore

$$\varepsilon_1 + \varepsilon_2 = (\alpha_1 + \alpha_2) + (\beta_1 + \beta_2).$$

By considering the triangle pairs CFS_1, CFS_2 and DGS_1, DGS_2, it can be shown that

$$\alpha = \alpha_1 + \alpha_2 \quad \text{and} \quad \beta = \beta_1 + \beta_2$$

from which it follows, as required, that

$$\varepsilon_1 + \varepsilon_2 = \alpha + \beta. \tag{1}$$

(Although the sign of the angles in equation 1 clearly varies according to the orientation of line and conic, a simple sign convention allows the relative signs to be readily determined.)

In the case of the parabola, as one of the foci is at infinity, one of the angles at the foci must be zero. The equation therefore reduces to $\varepsilon = \alpha + \beta$. In the case of the circle, let $\varepsilon_1 = \varepsilon_2 = \varepsilon$. The equation then reduces to $2\varepsilon = \alpha + \beta$.

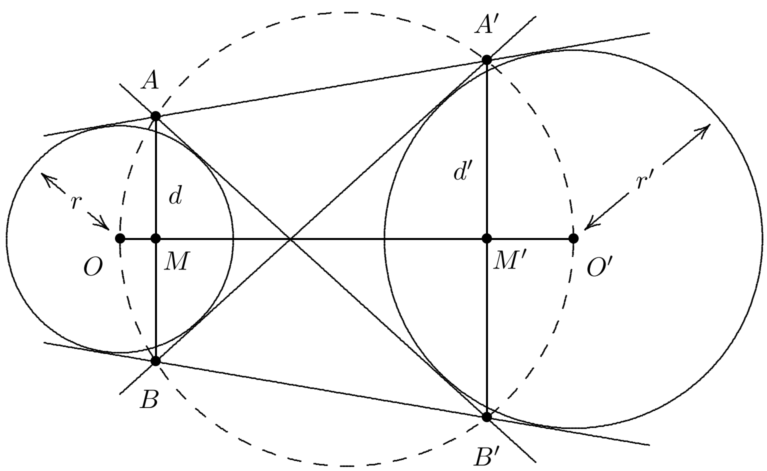

Figure 2:

Now applying this theorem to two circles facilitates the following ‘six-point’ circle proof (see Figure 2). Let O and O' be the centres of circles whose respective radii are r and r'. Let the internal and external tangents intersect in A, A', B, B'. As AB and $A'B'$ are perpendicular to OO', then $AA'B = AB'B$. It follows from the above result that $AA'B + AB'B = 2AO'B$ and therefore $AA'B = AB'B = AO'B$. Similarly $A'AB' = A'BB' = A'OB'$ and so the two centres O and O' lie on the circle A, A', B, B'. Therefore the six points A, A', B, B', O, O' are concyclic. Furthermore, if $AB = 2d$ and $A'B' = 2d'$, it follows that $rr' = dd'$. The details of the proof of this result are left to the interested reader.

Finally, the following locus property which has an interesting visual correlate, follows directly from the above results. Consider the circle ABO in Figure 3, where M is the midpoint of AB, and $AB \perp MO$.

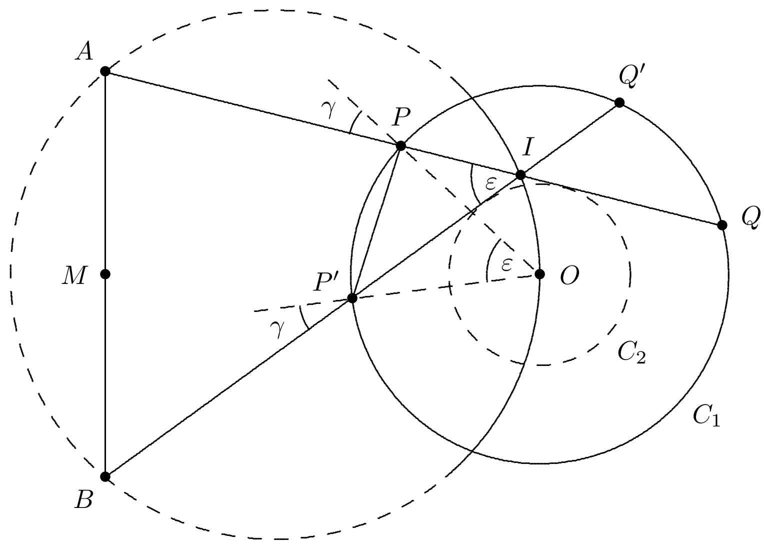

Figure 3:

THEOREM. *If PP' is a variable chord of any circle (C_1) subtending the same constant angle at the centre (O) as that subtended by AB at O, then the locus (I) of the intersection of AP and BP' is an arc of the circle ABO.*

PROOF. Construct any circle (C_2) smaller than and concentric with C_1 and let tangents from A and B to the circle C_2 intersect in I and cut C_1 in P and P', all as shown in Figure 3. As an easy consequence of of the earlier result, angles AIB and AOB are equal and so I lies on the circle ABO. By the same argument, $POP' = PIP'$, and therefore $POP' = AOB$. From this it follows that $QOQ' = POP'$, and therefore the intersection I lies on the circle ABO (irrespective of whether the chord is at PP' or QQ'). This holds for all circles C_2, and so the locus I is an arc of the circle ABO for all positions of the chord PP'.

Furthermore, since P, I, O, P' are concyclic (as $PIP' = POP'$) it follows that $IPO = IP'O$. Thus the lines AP and BP' meet the circle C_1 with the same angle of incidence γ.

Interestingly, the extent of the locus depends on the position of the chord AB in relation to the circle C_1, being a simple arc if AB is outside C_1. However, if AB is a chord of C_1, then one revolution of PP' results in I making one complete revolution of ABO. If AB is inside C_1, then the locus is two revolutions of ABO.

A visual correlate (the Pulfrich effect)

Under certain circumstances it is possible to actually see the locus of the point I as the path of an object which appears to move back and forth along an arc of the circle ABO, in a striking visual illusion.

Consider the eyes to be at the points A and B, viewing an object P rotating clockwise with constant angular velocity (ω) in a horizontal plane about a centre O. If P is viewed horizontally and binocularly with a neutral density filter in front of the right eye (B), and with AB perpendicular to MO, then under certain conditions of viewing distance (MO), angular velocity (ω), and filter density (i.e. when $POP' = AOB$), the object P will seem not to be rotating about O, but will appear to be moving back and forth from side to side, passing through the centre O and describing an arc of the circle ABO.

The illusion is due to the fact that the filter introduces a slight visual time delay (say, t) which is therefore equivalent to the angle POP' ($= \omega t$). The filtered eye therefore, always sees the object where it was a few milliseconds ago (i.e. at P'). It is thought that fusion of these two images results in the object P appearing to be at the position I.

This illusion is a particular manifestation of a little known phenomenon called the *Pulfrich effect*[2], after Carl Pulfrich (1858–1927) who first investigated it at the Carl Zeiss laboratories in Jena. The geometry of this circular configuration gives an unusual way of considering this effect.

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[2]Pulfrich, C. (1922). Die Stereoskopie im Dienste der isochromen und heterochromen Photometrie. *Naturwissenschaften*, 10, 553–564.
Nickalls RWD (1986). The rotating Pulfrich effect, and a new method of determining visual latency differences. *Vision Research*; 26, 367–372.
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